1.4 – Inverses; Algebraic Properties of Matrices

Theorem 1.4.1 Properties of Matrix Arithmetic
Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid for matrices A, B, and C and scalars a, b, and c.

a) $A + B = B + A$ (commutative law for matrix addition)
b) $A + (B + C) = (A + B) + C = A + B + C$ (associative law for matrix addition)
c) $(AB)C = A(BC) = ABC$ (associative law for matrix multiplication)
d) $A(B + C) = AB + AC$ (left distributive law)
e) $(B + C)A = BA + CA$ (right distributive law)
f) $A(B - C) = AB - AC$
g) $(B - C)A = BA - CA$
h) $a(B + C) \boxed{=} aB + aC$
i) $a(B - C) = aB - aC$
j) $(a + b)C = aC + bC$
k) $(a - b)C = aC - bC$
l) $a(bC) = (ab)C$
m) $a(BC) = (aB)C = B(aC)$

pf: (h) For $a(B+C)$ to equal $aB+aC$, they must have the same size and equal entries. If $B+C$ is defined (as assumed), they have the same size, say $m \times n$. Scalar mult. does not change the size of a matrix, so aB and aC are also both $m \times n$ so is their sum.
$(a(B+C))_{ij}$: is the entry in the i^{th} row and j^{th} column of the matrix on the left-hand side.

By the distributive property of real #s and the definitions of matrix addition and scalar mult., $(a(B+C))_{ij} = a(B+C)_{ij}$ ← position

$= a(b_{ij} + c_{ij})$ ← real #s

$= ab_{ij} + ac_{ij} = (aB)_{ij} + (aC)_{ij}$

Since corresponding entries are equal, $a(B+C) = aB + aC$.

Definition: In general, $AB \neq BA$. In the special cases where $AB = BA$, we say that A and B **commute**.

Definition: A **zero matrix**, denoted O, is a matrix whose entries are all zero.

Theorem 1.4.2 Properties of Zero Matrices
If c is a scalar, and if the sizes of the matrices A and O are such that the operations can be performed, then:
a) $A + O = O + A = A$
b) $A - O = A$
c) $A - A = A + (-A) = O$
d) $OA = O$
e) If $cA = O$, then $c = 0$ or $A = O$

However if $AC = BC$, it does not follow that $A = B$. Also, we can have $A \neq O$ and $B \neq O$ but $AB = O$.

Definition: Identity matrices are square matrices with 1's on the main diagonal and zeros everywhere else. They are denoted I or I_n if referencing the size, $n \times n$.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$AI = IA = A$ $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

square

Theorem 1.4.3 If R is the reduced row echelon form of an $n \times n$ matrix A, then either R has at least one row of zeros or R is the identity matrix I_n.

pf: An $n \times n$ matrix has at most n pivots. If there are n pivots, then $R = I_n$ by definition of rref. If there are fewer than n pivots, then at least one row does not have a leading 1 and so is a row of zeros, again by def of rref.

Definition: If A is a square matrix, and if there exists a matrix B of the same size for which $AB = BA = I$, then A is said to be **invertible** (or **nonsingular**) and B is called the **inverse** of A. If no such matrix B exists, then A is said to be **singular**. Because the inverse of a matrix A is unique, we will denote it using A^{-1}.

$AA^{-1} = A^{-1}A = I$

Theorem 1.4.4 Uniqueness of a Matrix Inverse
If B and C are both inverses of the matrix A, then $B = C$.

pf: Recall that $AI = IA = A$ by def of I.

$$B \underset{\text{Def of } I}{=} BI = \underset{C \text{ is an inverse of } A}{B(AC)} \underset{\text{associative prop of matrix mult.}}{=} (BA)C \underset{B \text{ is an inverse of } A}{=} IC \underset{\text{Def of } I}{=} C$$

Theorem 1.4.5 The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

#8 Use Theorem 1.4.5 to compute the inverse.

$\begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix} = A$

$\frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = I$ ✓

$$A^{-1} = \frac{1}{-6+8}\begin{bmatrix} -1 & -4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} -1/2 & -2 \\ 1 & 3 \end{bmatrix}$$

check: $\begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -1/2 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Theorem 1.4.6 If A and B are invertible matrices with the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

~~$(AB)^{-1} = A^{-1}B^{-1}$~~

pf: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ (assoc property of matrix mult.)

$= AIA^{-1}$ (def of B^{-1})

$= AA^{-1}$ (def of I)

$= I$ (def of A^{-1})

Since multiplying (AB) by $(B^{-1}A^{-1})$ yields I,
$B^{-1}A^{-1} = (AB)^{-1}$.

Theorem 1.4.7 Properties of Negative Exponents

If A is invertible and n is a nonnegative integer, then:

a) A^{-1} is invertible and $\left(A^{-1}\right)^{-1} = A$.

b) A^n is invertible and $(A^n)^{-1} = A^{-n} = \left(A^{-1}\right)^n$.

c) kA is invertible for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$.

reciprocal

inverse

#15 Use the given information to find A.

$$\left((7A)^{-1}\right)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}^{-1} \Rightarrow 7A = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} -2 & -7 \\ -1 & -3 \end{bmatrix}$$

$\frac{1}{ad-bc}$

$$7A = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 2/7 & 1 \\ 1/7 & 3/7 \end{bmatrix}$$

OR $\frac{1}{7}A^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} -21 & 49 \\ 7 & -14 \end{bmatrix} \ldots$

#23 Find all values of a, b, c, and d (if any) for which the matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ commute.

A & B commute if $AB = BA$.

$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$

$BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$ if $\boxed{c = 0,\ a = d \quad b \text{ is any number}}$

#25 Solve the system using an inverse matrix.

$3x_1 - 2x_2 = -1$

$4x_1 + 5x_2 = 3$

Matrix equation $\begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\cancel{A^{-1}A}\,\vec{x} = A^{-1}\vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$

$A^{-1} = \frac{1}{23}\begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{23}\begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{23}\begin{bmatrix} 1 \\ 13 \end{bmatrix}$

$(x_1, x_2) = \left(\frac{1}{23}, \frac{13}{23}\right)$

#**39** Simplify the expression assuming that A, B, C and D are invertible.
$(AB)^{-1}\left(AC^{-1}\right)\left(D^{-1}C^{-1}\right)^{-1}D^{-1}$

$B^{-1}\underbrace{A^{-1}A}_{I}\underbrace{C^{-1}C}_{I}\underbrace{DD^{-1}}_{I} = B^{-1}$

Theorem 1.4.8 Properties of the Transpose
If the sizes of the matrices are such that the stated operations can be performed, then:
a) $\left(A^T\right)^T = A$
b) $(A+B)^T = A^T + B^T$
c) $(A-B)^T = A^T - B^T$
d) $(kA)^T = kA^T$
e) $(AB)^T = B^T A^T$

A - matrix
$[a_{ij}]$ - matrix
$(A)_{ij}$ - entry (position)
a_{ij} - entry (scalar)

Pf: (b) Let $A = [a_{ij}]$ and $B = [b_{ij}]$ matrices of the same size.

$((A+B)^T)_{ij} = (A+B)_{ji} = (A)_{ji} + (B)_{ji}$

by def of transpose

$= (A^T)_{ij} + (B^T)_{ij} = (A^T + B^T)_{ij}$

Thus $(A+B)^T = A^T + B^T$

Theorem 1.4.9 If A is an invertible matrix, then A^T is also invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

Pf: Since A is invertible A^{-1} exists.

$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$

Since $A^T(A^{-1})^T = I$, $(A^{-1})^T = (A^T)^{-1}$ ✓

Pf:

Consider AB. This is defined if A is $m \times r$ and B is $r \times n$.

$$(AB)_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}$$

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_{k=1}^{r} a_{jk} b_{ki}$$

$$(B^T A^T)_{ij} = \sum_{k=1}^{r} (B^T)_{ik} (A^T)_{kj} = \sum_{k=1}^{r} b_{ki} a_{jk}$$

Note that $\sum_{k=1}^{r} a_{jk} b_{ki} = \sum_{k=1}^{r} b_{ki} a_{jk}$ by comm. prop of real #s.

So $(AB)^T = B^T A^T$ because the entries are equal.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

$$(AB)_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ir} b_{rj}$$

position

#s inside matrix

Thm 1.4.8 d) $(AB)^T = B^T A^T$.

Pf: Let $((AB)^T)_{ij}$ be the entry in the i^{th} row and j^{th} column of $(AB)^T$.

Then $((AB)^T)_{ij} = (AB)_{ji}$, the sum of the products of the entries in the j^{th} row of A and the i^{th} column of B.

These are exactly the entries in the i^{th} row of B^T and j^{th} column of A^T.

Thus $((AB)^T)_{ij} = (B^T A^T)_{ij}$, so

$$(AB)^T = B^T A^T.$$